

# TRANSLATING SOLUTIONS TO THE GAUSS CURVATURE FLOW WITH FLAT SIDES

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**ABSTRACT.** We derive local  $C^2$  estimates for complete non-compact translating solitons of the Gauss curvature flow in  $\mathbb{R}^3$  which are graphs over a convex domain  $\Omega$ . This is closely related to deriving local  $C^2$  estimates for the degenerate Monge-Ampère equation. As a result, in the special case where the domain  $\Omega$  is a square, we establish the existence of a  $C_{\text{loc}}^{1,1}$  translating soliton with flat sides. The existence of flat sides follows from a local a priori non-degeneracy estimate near the free-boundary.

## 1. INTRODUCTION

We recall that an one-parameter family of immersions  $F : M^n \times (0, T) \rightarrow \mathbb{R}^{n+1}$  is a solution of the Gauss curvature flow, if for each  $t \in (0, T)$ ,  $F(M^n, t) = \Sigma_t$  is a complete convex hypersurface embedded in  $\mathbb{R}^{n+1}$  satisfying

$$(1.1) \quad \frac{\partial}{\partial t} F(p, t) = K(p, t) \vec{n}(p, t)$$

where  $K(p, t)$  and  $\vec{n}(p, t)$  are the Gauss curvature and the interior unit vector of  $\Sigma_t$  at the point  $F(p, t)$ , respectively.

In this paper, we consider a translating solution  $\Sigma_t$  to the Gauss curvature flow in  $\mathbb{R}^3$  satisfying

$$\Sigma_t = \Sigma + ct \vec{e}_3 =: \{Y + ct \vec{e}_3 \in \mathbb{R}^3 : Y \in \Sigma\},$$

where  $\vec{e}_3 = (0, 0, 1)$  and the *speed*  $c$  is a constant, and  $\Sigma$  is a complete convex hypersurface embedded in  $\mathbb{R}^3$ . We observe that there exist a convex open set  $\Omega \subset \mathbb{R}^2$  and a convex function  $u : \Omega \rightarrow \mathbb{R}$  satisfying

$$\Sigma \text{ is the boundary of } \{(x, t) : x \in \Omega, t \geq u(x)\}.$$

By the result in [20], the set  $\Omega$  must be *bounded*. If  $\mathcal{A}(\Omega)$  denotes the area of  $\Omega$ , it follows that  $u$  is a smooth function satisfying

$$(1.2) \quad \begin{cases} \frac{\det D^2 u}{(1 + |Du|^2)^{\frac{3}{2}}} = \frac{2\pi}{\mathcal{A}(\Omega)} & \text{in } \Omega, \\ \lim_{x \rightarrow \partial\Omega} |Du|(x) = +\infty & \text{on } \partial\Omega. \end{cases}$$

Conversely, given an open bounded convex set  $\Omega \subset \mathbb{R}^2$ , there exists a solution  $u : \Omega \rightarrow \mathbb{R}$  of (1.2), and any two solutions differ by a constant. (See [20] and Theorem 4.8 in [19]). Hence, given a translator  $\Sigma$  in  $\mathbb{R}^3$  of the Gauss curvature flow, there exists an open bounded convex set  $\Omega \in \mathbb{R}^2$  such that  $\Sigma$  converges to the cylinder  $\partial\Omega \times \mathbb{R}$ , and the immersion  $F : M^2 \rightarrow \mathbb{R}^3$  of  $F(M^2) = \Sigma$  satisfies

$$(*) \quad K(p) = \frac{2\pi}{\mathcal{A}(\Omega)} \langle \vec{n}(p), \vec{e}_3 \rangle.$$

We recall the result of John Urbas in [20].

**Theorem 1.1** (Urbas). *Given an open bounded convex domain  $\Omega \subset \mathbb{R}^2$ , there exists a convex solution  $u : \Omega \rightarrow \mathbb{R}$  satisfying (1.2), and it is unique up to addition by a constant. In particular, if for each  $x_0 \in \partial\Omega$ , there exists a ball  $B \subset \mathbb{R}^2$  satisfying  $\Omega \subset B$  and  $x_0 \in \partial B$ , then the solution  $u$  is a smooth function satisfying*

$$\lim_{x \rightarrow \partial\Omega} u(x) = +\infty.$$

This result guarantees that there exists a unique  $C^1$  translator  $\Sigma = \partial\{(x, t) : x \in \Omega, t \geq u(x)\}$  for any open bounded convex domain  $\Omega$ . Also, if  $\Omega$  is a *uniformly convex* domain, then  $\Sigma$  is *strictly convex*, and thus  $C^\infty$  smooth by standard estimates. However, if  $\Omega$  is weakly convex, then  $\Sigma$  may not be strictly convex on the boundary of  $\Omega$ . Richard Hamilton conjectured that if  $\Omega$  is a *square*, then  $\Sigma$  has *flat sides* on the boundary of  $\Omega$ . This is shown in the next picture.

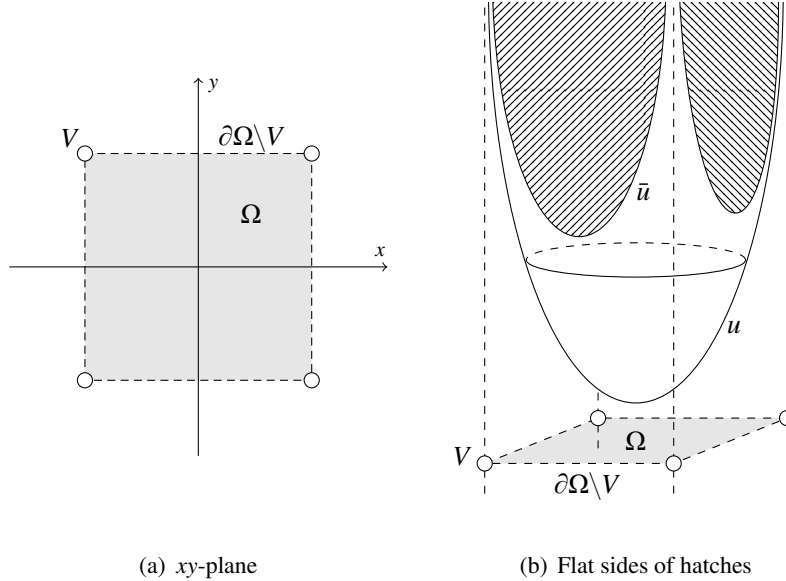


FIGURE 1. Translator  $\Sigma$  on a square

The main result in this paper proves Hamilton's conjecture and establishes the *optimal regularity* of the translator in this degenerate case.

**Theorem 1.2.** *Let  $\Omega := (-1, 1) \times (-1, 1) \subset \mathbb{R}^2$  be the open square and  $V$  be the set of the vertices  $(\pm 1, \pm 1)$  of  $\partial\Omega$ . Assume  $u : \Omega \rightarrow \mathbb{R}$  is a convex smooth solution of (1.2), and let  $\Sigma$  denote the boundary of  $\{(x, t) : x \in \Omega, t \geq u(x)\}$ . Then,  $\Sigma$  is a complete convex hypersurface of class  $C_{loc}^{1,1}$ , and there exists a smooth function  $\bar{u} : (\partial\Omega \setminus V) \rightarrow \mathbb{R}$  satisfying*

$$\lim_{x \rightarrow y} u(x) = \bar{u}(y), \quad \lim_{x \rightarrow V} u(x) = \lim_{y \rightarrow V} \bar{u}(y) = +\infty.$$

The Gauss curvature flow was first introduced by W. Firey in [13], where he showed that a closed strictly convex and centrally symmetric solution in  $\mathbb{R}^3$  converges to a round point. In [18] K. Tso established the existence of closed and strictly convex solutions in  $\mathbb{R}^{n+1}$  and showed that it converges to a point. B. Andrews [2] extended Tso's result to the flow by positive powers of the Gauss curvature, namely a strictly convex closed solution, to the  $\alpha$ -Gauss curvature flow  $\partial_t F = K^\alpha \vec{n}$ .

The convergence of the flow to a closed self-similar solution has been widely studied in [1, 2, 3, 4, 9, 16, 14] for  $\alpha > \frac{1}{n+2}$ . In the case  $\alpha = \frac{1}{n+2}$ , E. Calabi [5] that showed that closed self-similar solutions are ellipsoids. In the case  $\alpha = \frac{1}{n}$ , B. Chow [9] established the convergence to the round sphere, namely the sphere is the unique closed self-similar solution. In the two dimension case  $n = 2$ , B. Andrews showed the convergence to sphere for  $\alpha = 1$  in [1], and for  $\alpha \in (\frac{1}{2}, 1)$  with X. Chen in [3]. In the higher dimensions  $n \geq 2$ , the uniqueness of closed self-similar solutions was shown for  $\frac{1}{n} < \alpha < 1 + \frac{1}{n}$  in [7].

Recently, the all-time existence of a non-compact complete and strictly convex solution  $\alpha$ -Gauss curvature flow in  $\mathbb{R}^{n+1}$  was established in [8]. Therefore, the convergence of such solutions to complete self-similar solutions becomes a natural question. Different from the closed case, self-expanders for  $\alpha > 0$  and translators  $\alpha > \frac{1}{2}$  have been completely classified by J. Urbas [20] in  $C^1$  sense. In this paper, we will improve the  $C^1$  regularity to  $C^{1,1}$  for  $n = 2$  and  $\alpha = 1$ , and we will show that the  $C^{1,1}$  is the optimal regularity. See Remark 1.3. As our main result states, we are especially interested in translators with flat sides.

Closed solutions of the Gauss curvature flow in  $\mathbb{R}^3$  with a flat sides was considered by R. Hamilton in [15], and the  $C^\infty$  regularity of its free boundary was studied in [10, 11, 17]. The optimal  $C^{1,1}$  regularity for  $n = 2$  and  $\alpha = 1$  was obtained in [1], and the  $C^{1,\beta}$  regularity for other  $n$  and  $\alpha$  was established in [12].

#### *Discussion on the Proof:*

To establish the interior  $C^{1,1}$  regularity of the surface  $\Sigma$  we will bound  $\eta \Lambda$ , where  $\Lambda = \max(\lambda_1(p), \lambda_2(p))$  denotes the largest principal curvature of  $\Sigma$  and  $\eta$  is the cut-off function  $\eta = (|F(p)|^2 - R^2)_+$  as defined in Notation 2.1. If one chooses instead the standard cut-off function  $\eta_s(p) = (R^2 - |F(p)|^2)_+$  or the level set cut-off function  $\eta_L(p) = (M - \langle \vec{e}_3, \vec{n}(p) \rangle)_+$ , then the estimate fails because of counter examples. Since the equation (\*) only depends on the area  $\mathcal{A}$ , the solution  $\Sigma_1$  defined on  $(0, 1) \times (0, 1)$  satisfies the same equation to the solution  $\Sigma_\epsilon$  defined on  $(0, \epsilon) \times (0, 1/\epsilon)$ . However, it is clear that  $\Sigma_\epsilon$  has larger curvature than  $\Sigma_1$  near the tip. However,  $\eta_s$  and  $\eta_L$  can not distinguish between  $\Sigma_1$  and  $\Sigma_\epsilon$ . Thus, one should use a cut-off function which included information about the global structure  $\Omega$  as the cut-off function  $\eta$  in Notation 2.1.

To show the existence of flat sides, we need to deal with technical difficulties arising from the non-compactness of our solution  $\Sigma$ . One of the difficulties is the double degenerate nature on the flat sides. We may consider  $(0, 1, 0) \in \mathbb{R}^3$  as the height vector, and define a convex function  $h(x, z)$  whose graph  $(x, h(x, z), z)$  is the lower part of the solution  $\Sigma$ . Then,  $h(x, z)$  satisfies

$$(1.3) \quad \frac{\det D^2 h}{(1 + |Dh|^2)^{\frac{3}{2}}} = -\frac{2\pi}{\mathcal{A}(\Omega)} D_z h.$$

Then, near the flat side  $\{(x, z) : h = -1\}$ , the right hand side  $-D_z h = |Dh| \langle -v, e_2 \rangle$  has two degenerate factors  $|Dh|$  and  $\langle -v, e_2 \rangle$ , where  $e_2 = (0, 1)$  and  $v \in \mathbb{R}^2$  is the outward normal direction of the level set of  $h$ . Since the level sets of  $h$  becomes parallel to  $e_2$  at the infinity,  $\langle -v, e_2 \rangle$  goes to zero at the infinity.

Another challenge comes from the fact that the non-degeneracy of  $|Dh|$  depends on the global structure  $\Omega$ . In the previous works [10, 11, 15, 17], the lower bound for  $|Dh| h^{-\frac{1}{2}}$  depends on the initial data, and this does not apply in the elliptic setting. Hence, we have to develop a new non-degeneracy estimate for a solution  $h$  to the horizontal equation (1.3) which includes the information of the global structure of the domain  $\Omega$ . We will consider the gradient bound of a solution  $u$  to (1.2) instead of the non-degeneracy of a solution  $h$  to (1.3) for convenience.

*Outline of the paper:*

A brief *outline* of this paper is as follows : In section 2 we will summarize the notation which will be used throughout the paper. In section 3, we will derive the interior  $C^2$  estimate for any strictly convex complete smooth solution  $\Sigma$  by using a Pogorelov type computation which we will localize by introducing the cutoff function  $\eta$  mentioned above. In section 4, we construct a barrier. Since a complete supersolution defined on  $\Omega'$  satisfies  $\mathcal{A}(\Omega') > \mathcal{A}(\Omega)$ , we can not put such a supersolution on top of a our solution  $\Sigma$ . Instead, we need to cut a complete supersolution and slide it to contact  $\Sigma$ . This way we obtain the gradient bound of the level set at the contact point, which leads to the local lower bound  $\langle -v, e_2 \rangle$ . This implies the crucial partial derivative estimate of a solution  $u$  to (1.2) in Theorem 4.4. In section 4, we derive a separable equation from the equation (1.2), namely

$$\frac{u_{xx}u_{yy}}{(1 + u_y^2)^{\frac{3}{2}}} \geq \frac{\det D^2 u}{(1 + |Du|^2)^{\frac{3}{2}}} = \frac{2\pi}{\mathcal{A}(\Omega)}.$$

By integrating this equation, we obtain the gradient bound at a certain point  $(x_0, -1 + \epsilon)$  in Lemma 5.1. Then, we establish the distance bound in Theorem 5.2.

*Remark 1.3* (Optimal regularity). Let us show that if there exists a flat side on a translator  $\Sigma$ , then  $\Sigma$  has at most  $C^{1,1}$  regularity. The horizontal equation (1.3) yields

$$h_{vv}h_{\tau\tau} \geq \frac{\det D^2 h}{(1 + |Dh|^2)^{\frac{3}{2}}} \geq -\frac{2\pi}{\mathcal{A}} |Dh| \langle v, e_2 \rangle$$

where  $v(x, z)$  is the outward normal and  $\tau(x, z)$  is a tangential direction of the level set of  $h(x, z)$  at a point  $(x, z)$ . We denote by  $L_r$  the level set  $\{(x, z) : h(x, z) = r\}$ , and denote by  $\kappa(x, z)$  the curvature of  $L_{h(x,z)}$  at  $(x, z)$ . Since we have  $h_{\tau\tau} = |Dh| \kappa$ , the inequality above gives  $h_{vv} \kappa \geq -2\pi \mathcal{A}^{-1} \langle v, e_2 \rangle$ . We will establish the local lower bound for  $-\langle v, e_2 \rangle$  in section 3, which guarantees that

$$h_{vv} \kappa \geq c.$$

We choose a neighborhood  $U$  of a point  $(x_0, z_0)$  on the free boundary  $\Gamma$ , namely  $(x_0, z_0) \in \Gamma =: \partial L_{-1}$ . Since the level sets  $L_r$  monotonically converge to  $\Gamma$ , there exists a constant  $c$  such that  $\int_{L_r \cap U} ds \geq c$  for  $r$  close enough to  $-1$ , where  $s$  is the arc length parameter. Hence, the following holds

$$2\pi \geq \int_{L_r} \kappa ds \geq \int_{L_r \cap U} \kappa ds \geq \frac{c}{\max_{L_r \cap U} h_{vv}}.$$

Thus,  $\max_{L_r \cap U} h_{vv} \geq c$  holds for some uniform constant  $c$ . However, we have  $D^2h = 0$  on  $L_{-1}$ . Therefore,  $D^2h$  is not a continuous function.

## 2. NOTATION

For the convenience of the reader, we give below some basic notation which will be frequently used in what follows. We will use Definition 2.2 and Notation 2.1 in section 3. Definition 2.3 and Notation 2.4 will be used in section 4 and 5.

**Notation 2.1** (To be used in section 3). We will use some standard notation on the metric, second fundamental form and the linearized operator.

- (i) Let  $F : M^2 \rightarrow \mathbb{R}^3$  be an immersion defining a smooth and complete surface  $\Sigma$  by  $F(M^2) = \Sigma$ . If  $B_R(Y)$  cut  $\Sigma$  as Definition 2.2, we define a cut-off function  $\eta : M^2 \rightarrow \mathbb{R}$  by

$$\eta(p) = (|F(p) - Y|^2 - R^2)_+.$$

- (ii) We recall that  $g_{ij} = \langle F_i, F_j \rangle$ , where  $F_i := \nabla_i F$ . Also, we denote as usual by  $g^{ij}$  the inverse matrix of  $g_{ij}$  and  $F^i = g^{ij} F_j$ .
- (iii) For a strictly convex smooth hypersurface  $\Sigma_t$ , we denote by  $b^{ij}$  the the inverse matrix  $(h^{-1})^{ij}$  of its *second fundamental form*  $h_{ij}$ , namely  $b^{ij} h_{jk} = \delta_k^i$ .
- (iv) We denote by  $\mathcal{L}$  the *linearized* operator

$$\mathcal{L} = Kb^{ij} \nabla_i \nabla_j.$$

Furthermore,  $\langle \cdot, \cdot \rangle_{\mathcal{L}}$  denotes the associated inner product  $\langle \nabla f, \nabla g \rangle_{\mathcal{L}} = Kb^{ij} \nabla_i f \nabla_j g$ , where  $f, g$  are differentiable functions on  $M^n$ , and  $\| \cdot \|_{\mathcal{L}}$  denotes the  $\mathcal{L}$ -norm given by the inner product  $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ .

- (v)  $H$  and  $\Lambda$  denote the *mean curvature* and the *largest principal curvature*, respectively.

**Definition 2.2** (Cutting ball). Given a ball  $B_R(Y) \subset \mathbb{R}^3$  and a complete surface  $\Sigma \subset \mathbb{R}^3$ , we say that a compact surface  $\Sigma_c$  with boundary  $\partial \Sigma_c$  is cut off from  $\Sigma$  by  $B_R(Y)$ , if  $\Sigma_c \subset \Sigma$  and  $\partial \Sigma_c \subset \partial B_R(Y)$  hold.

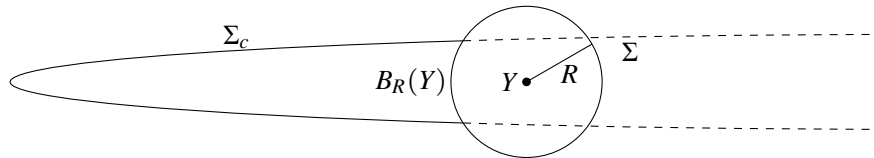


FIGURE 2. Cutting ball

**Definition 2.3** (Axial symmetry). We say that a surface  $\Sigma \subset \mathbb{R}^3$  has *axial symmetry*, if  $(x, y, z) \in \Sigma$  guarantees  $(-x, y, z), (x, -y, z) \in \Sigma$ . Similarly, a set  $\Omega \subset \mathbb{R}^2$  has *axial symmetry*, if  $(x, y) \in \Omega$  guarantees  $(-x, y), (x, -y) \in \Omega$ .

**Notation 2.4** (To be used in sections 4 and 5). Also, we summarize some further notation.

- (i) Given a set  $A \subset \mathbb{R}^3$  and a constant  $s$ , we denote the  $x = s$  level set by  $L_s^x(A) = \{(s, y, z) \in A\}$ . Similarly, we denote  $y = s$  and  $z = s$  level set by  $L_s^y(A)$  and  $L_s^z(A)$ , respectively.
- (ii) Given a constant  $s$  and a function  $f : \Omega \rightarrow \mathbb{R}$  with  $\Omega \subset \mathbb{R}^2$ , we denote by  $L_s(f)$  the  $s$ -level set  $\{(x, y) \in \Omega : f(x, y) = s\}$ .
- (iii) We let  $e_1$  and  $e_2$  the unit vectors  $(1, 0)$  and  $(0, 1)$ , respectively.
- (iv) For a complete and convex curve  $\Gamma \subset \mathbb{R}^2$ , its convex hull  $\text{Conv}(\Gamma)$  is given by

$$\text{Conv}(\Gamma) = \{(tx + (1 - t)y : x, y \in \Gamma, t \in [0, 1]\}.$$

If  $A$  is a subset of  $\text{Conv}(\Gamma)$ , then we say  $A$  is enclosed by  $\Gamma$  and use the notation

$$A < \Gamma.$$

- (v) Given a set  $A \subset \mathbb{R}^2$ ,  $\text{cl}(A)$  and  $\text{Int}(A)$  mean the closure and the interior of  $A$ , respectively.

### 3. OPTIMAL $C^{1,1}$ REGULARITY

In this section, we will establish a local curvature estimate for smooth strictly convex complete solutions of equation (\*). In the last section we will use this estimate to obtain the optimal  $C^{1,1}$  regularity for a weakly convex solution of (\*) in the degenerate case. We recall that a solution of (\*) has an immersion  $F : M^2 \rightarrow \mathbb{R}^3$  of  $F(M^2) = \Sigma$ . Given a ball  $B_R(Y)$  we define the associated cut-off function  $\eta$  by  $\eta(p) = (|F(p) - Y|^2 - R^2)_+$ . We have the following result.

**Theorem 3.1** (Curvature bound). *Let  $\Sigma$  be a smooth strictly convex complete solution of (\*). Let  $\Sigma_c$  be the cut off from  $\Sigma$  by a ball  $B_R(Y) \subset \mathbb{R}^3$  as defined in Definition 2.2. Then, for any  $p \in M^2$  with  $F(p) \in \Sigma_c$ , the maximum principal curvature  $\Lambda(p) := \max\{\lambda_1(p), \lambda_2(p)\}$  satisfies*

$$\eta\Lambda(p) \leq \frac{9\pi}{\mathcal{A}(\Omega)} \sup_{F(q) \in \Sigma_c} |F(q) - Y|^3.$$

*Proof.* We may assume, without loss of generality, that  $Y = 0$ . Recall the definition of the cutting ball as shown in Figure 2 above. The continuous function  $\eta\Lambda$  attains its maximum on the compact set  $\Sigma_c$  at some point  $F(p_0) \in \Sigma_c$ ,

$$\eta\Lambda(p_0) = \max_{F(p) \in \Sigma_c} \eta\Lambda(p).$$

Then, because we have  $\eta = 0$  on  $\partial\Sigma_c$ ,  $F(p_0)$  is an interior point of  $\Sigma_c$ . Thus,  $\eta\Lambda$  attains a local maximum at  $p_0$ . Moreover, we can choose an open chart  $(U, \varphi)$  with  $p_0 \in \varphi(U)$  and  $F(\varphi(U)) \subset \Sigma_c$  such that the covariant derivatives  $\{\nabla_1 F(p_0), \nabla_2 F(p_0)\}$  form an orthonormal basis of  $T\Sigma_{F(p_0)}$  satisfying

$$g_{ij}(p_0) = \delta_{ij}, \quad h_{ij}(p_0) = \delta_{ij}\lambda_i(p_0), \quad \lambda_1(p_0) = \Lambda(p_0).$$

Next, we define the function  $w : U \rightarrow \mathbb{R}$  by

$$w = \eta \frac{h_{11}}{g_{11}}.$$

Then, the Euler formula guarantees  $w \leq \eta\Lambda$  (c.f. Proposition 4.1 in [6]). Therefore, for all  $p \in U$ , the following holds

$$w(p) \leq \eta\Lambda(p) \leq \eta\Lambda(p_0) = w(p_0).$$

Thus,  $w$  also attains its maximum at  $p_0$ .

Now, we consider the derivative of  $w$ . Then,  $\nabla g_{11} = 0$  gives

$$(3.1) \quad \frac{\nabla_i w}{w} = \frac{\nabla_i h_{11}}{h_{11}} + \frac{\nabla_i \eta}{\eta}.$$

Differentiating the equation above yields

$$\frac{\nabla_i \nabla_j w}{w} - \frac{\nabla_i w \nabla_j w}{w^2} = \frac{\nabla_i \nabla_j h_{11}}{h_{11}} - \frac{\nabla_i h_{11} \nabla_j h_{11}}{(h_{11})^2} + \frac{\nabla_i \nabla_j \eta}{\eta} - \frac{\nabla_i \eta \nabla_j \eta}{\eta^2}$$

and multiplying by  $Kb^{ij}$ , we obtain

$$(3.2) \quad \frac{\mathcal{L}w}{w} - \frac{\|\nabla w\|_{\mathcal{L}}^2}{w^2} = \frac{\mathcal{L}h_{11}}{h_{11}} - \frac{\|\nabla h_{11}\|_{\mathcal{L}}^2}{(h_{11})^2} + \frac{\mathcal{L}\eta}{\eta} - \frac{\|\nabla \eta\|_{\mathcal{L}}^2}{\eta^2}.$$

Observing next that  $\mathcal{L}F := Kb^{ij} \nabla_j \nabla_j F = Kb^{ij} h_{ij} \vec{n} = 2K \vec{n}$ , we compute  $\mathcal{L}\eta$  on the support of  $\eta$  as follows:

$$\mathcal{L}\eta = \mathcal{L}|F|^2 = 2\langle F, \mathcal{L}F \rangle + 2\langle \nabla F, \nabla F \rangle_{\mathcal{L}} = 4K\langle F, \vec{n} \rangle + 2Kb^{ij} g_{ij} = 4K\langle F, \vec{n} \rangle + 2H.$$

Thus,  $4K = 8\pi\mathcal{A}^{-1}\langle \vec{e}_3, \vec{n} \rangle \leq 8\pi\mathcal{A}^{-1}$  and  $2H \geq 2\Lambda$  imply

$$(3.3) \quad \mathcal{L}\eta \geq -8\pi\mathcal{A}^{-1}|F| + 2\Lambda.$$

Since  $w$  attains its maximum at  $p_0$ , we have  $\nabla w(p_0) = 0$ , and thus (3.1) gives

$$\frac{\|\nabla h_{11}\|_{\mathcal{L}}^2}{(h_{11})^2}(p_0) = \frac{\|\nabla \eta\|_{\mathcal{L}}^2}{\eta^2}(p_0).$$

Hence, combining  $\mathcal{L}w(p_0) \leq 0$ , (3.2), (3.3) and the equation above yields the following at  $p_0$

$$(3.4) \quad 0 \geq \frac{\mathcal{L}h_{11}}{h_{11}} + \frac{2\Lambda}{\eta} - \frac{8\pi|F|}{\mathcal{A}\eta} - \frac{2\|\nabla h_{11}\|_{\mathcal{L}}^2}{(h_{11})^2}.$$

To compute  $\mathcal{L}h_{11}$ , we begin by differentiating  $K$ ,

$$(3.5) \quad \nabla_1 K = Kb^{ij} \nabla_1 h_{ij}.$$

By differentiating the equation above again, we obtain

$$(3.6) \quad \nabla_1 \nabla_1 K = Kb^{ij} \nabla_1 \nabla_1 h_{ij} + Kb^{ij} b^{kl} \nabla_1 h_{ij} \nabla_1 h_{kl} - Kb^{ik} b^{jl} \nabla_1 h_{ij} \nabla_1 h_{kl}.$$

We can derive  $\mathcal{L}h_{11}$  from the first term  $Kb^{ij} \nabla_1 \nabla_1 h_{ij}$  as follows

$$(3.7) \quad \begin{aligned} Kb^{ij} \nabla_1 \nabla_1 h_{ij} &= Kb^{ij} \nabla_1 \nabla_i h_{j1} = Kb^{ij} (\nabla_i \nabla_1 h_{j1} + R_{1ijk} h_1^k + R_{1ikj} h_j^k) \\ &= Kb^{ij} \nabla_i \nabla_j h_{11} + Kb^{ij} (h_{1j} h_{ik} - h_{1k} h_{ij}) h_1^k + Kb^{ij} (h_{11} h_{ik} - h_{1k} h_{i1}) h_j^k \\ &= \mathcal{L}h_{11} - 2Kh_{1k} h_1^k + KHh_{11}. \end{aligned}$$

On the other hand, differentiating (\*) yields

$$(3.8) \quad \nabla_1 K = \frac{2\pi}{\mathcal{A}} \langle \nabla_1 \vec{n}, \vec{e}_3 \rangle = -\frac{2\pi}{\mathcal{A}} h_{1k} \langle F^k, \vec{e}_3 \rangle.$$

To get the right hand side of (3.6), we differentiate the equation above,

$$(3.9) \quad \nabla_1 \nabla_1 K = -\frac{2\pi}{\mathcal{A}} \nabla_1 h_{1k} \langle F^k, \vec{e}_3 \rangle - \frac{2\pi}{\mathcal{A}} h_{1k} h_1^k \langle \vec{n}, \vec{e}_3 \rangle = -\frac{2\pi}{\mathcal{A}} \nabla_k h_{11} \langle F^k, \vec{e}_3 \rangle - K h_{1k} h_1^k.$$

Combining (3.6), (3.7), and (3.9), we obtain the following at  $p_0$

$$\mathcal{L} h_{11} = 2|\nabla_2 h_{11}|^2 - 2\nabla_1 h_{11} \nabla_1 h_{22} - \frac{2\pi}{\mathcal{A}} \nabla_k h_{11} \langle F^k, \vec{e}_3 \rangle - K^2.$$

Hence, at  $p_0$ , applying the equation above to (3.4) and the definition of the norm  $\|\cdot\|_{\mathcal{L}}^2$  yield

$$\begin{aligned} 0 &\geq \frac{1}{h_{11}} (2|\nabla_2 h_{11}|^2 - 2\nabla_1 h_{11} \nabla_1 h_{22} - \frac{2\pi}{\mathcal{A}} \nabla_k h_{11} \langle F^k, \vec{e}_3 \rangle - K^2) \\ &\quad - \frac{2h_{22}|\nabla_1 h_{11}|^2 + 2h_{11}|\nabla_2 h_{11}|^2}{(h_{11})^2} + \frac{2\Lambda}{\eta} - \frac{8\pi|F|}{\mathcal{A}\eta} \\ &= -\frac{2\nabla_1 h_{11}(h_{22}\nabla_1 h_{11} + h_{11}\nabla_1 h_{22})}{(h_{11})^2} + \frac{1}{h_{11}} \left( -\frac{2\pi}{\mathcal{A}} \nabla_k h_{11} \langle F^k, \vec{e}_3 \rangle - K^2 \right) + \frac{2\Lambda}{\eta} - \frac{8\pi|F|}{\mathcal{A}\eta}. \end{aligned}$$

However, (3.5) and (3.8) imply the following at  $p_0$

$$h_{22}\nabla_1 h_{11} + h_{11}\nabla_1 h_{22} = \nabla_1 K = -\frac{2\pi}{\mathcal{A}} h_{11} \langle F^1, \vec{e}_3 \rangle.$$

Therefore, the last inequality can be reduced to

$$0 \geq -\frac{2\pi}{\mathcal{A}} \frac{\nabla_2 h_{11}}{h_{11}} \langle F^2, \vec{e}_3 \rangle + \frac{2\pi}{\mathcal{A}} \frac{\nabla_1 h_{11}}{h_{11}} \langle F^1, \vec{e}_3 \rangle - \frac{K^2}{h_{11}} + \frac{2\Lambda}{\eta} - \frac{8\pi|F|}{\mathcal{A}\eta}.$$

Observing  $(h_{11})^{-1} \nabla_i h_{11}(p_0) = -\eta^{-1} \nabla_i \eta(p_0)$  by (3.1) and  $\nabla w(p_0) = 0$ , we have

$$0 \geq \frac{2\pi}{\mathcal{A}} \frac{\nabla_2 \eta}{\eta} \langle F^2, \vec{e}_3 \rangle - \frac{2\pi}{\mathcal{A}} \frac{\nabla_1 \eta}{\eta} \langle F^1, \vec{e}_3 \rangle - \frac{K^2}{h_{11}} + \frac{2\Lambda}{\eta} - \frac{8\pi|F|}{\mathcal{A}\eta}.$$

Applying  $h_{11}(p_0) = \Lambda(p_0)$  and (\*) to the inequality above, we obtain

$$0 \geq \frac{4\pi}{\mathcal{A}\eta} \langle F_2, F \rangle \langle F^2, \vec{e}_3 \rangle - \frac{4\pi}{\mathcal{A}\eta} \langle F_1, F \rangle \langle F^1, \vec{e}_3 \rangle - \frac{4\pi^2 |\langle \vec{n}, \vec{e}_3 \rangle|^2}{\mathcal{A}^2 \Lambda} + \frac{2\Lambda}{\eta} - \frac{8\pi|F|}{\mathcal{A}\eta}.$$

We next multiply by  $\eta\Lambda$  the last inequality and apply  $|\langle F_i, F \rangle \langle F^i, \vec{e}_3 \rangle| \leq |F|$  and  $\langle \vec{n}, \vec{e}_3 \rangle \leq 1$ . Then, by also using the definition  $\eta = (|F|^2 - R^2)$ , we obtain

$$0 \geq -\frac{16\pi}{\mathcal{A}} |F| \Lambda - \frac{4\pi^2 \eta}{\mathcal{A}^2} + 2\Lambda^2 \geq 2\Lambda^2 - \frac{16\pi}{\mathcal{A}} |F| \Lambda - \frac{4\pi^2}{\mathcal{A}^2} |F|^2.$$

Solving the quadratic inequality of  $\Lambda$ , we obtain an upper bound of  $\Lambda$  at  $p_0$ ,

$$\Lambda \leq \frac{(4 + 3\sqrt{2})\pi|F|}{\mathcal{A}} \leq \frac{9\pi}{\mathcal{A}} |F|.$$

Therefore, multiplying by  $\eta \leq |F|^2$  yields the desired result,

$$\eta\Lambda(p) \leq \eta\Lambda(p_0) \leq \frac{9\pi}{\mathcal{A}} |F|^3(p_0) \leq \frac{9\pi}{\mathcal{A}} \sup_{F(q) \in \Sigma_c} |F|^3(q).$$



□

## 4. PARTIAL DERIVATIVE BOUND

Assume that  $\Omega$  is an open bounded strictly convex and smooth domain of  $\mathbb{R}^2$  and assume in addition that  $\Omega$  is axially symmetric. Let  $u$  be the unique solution of (1.2) on  $\Omega$  which defines the surface  $\Sigma$ . The following simple property readily follows from the uniqueness of solutions.

**Proposition 4.1** (Symmetry of solutions). *Let  $\Omega$  be an open bounded strictly convex and smooth subset of  $\mathbb{R}^2$  which is axially symmetric. Then, a solution  $u$  of (1.2) also is axially symmetric.*

The symmetry of  $u$  implies that the half surface  $\{(x, y, z) \in \Sigma : y \leq 0\}$  is the graph of a function  $h : \Omega_y \rightarrow \mathbb{R}$ , that is

$$\{(x, y, z) \in \Sigma : y \leq 0\} = \{(x, h(x, z), z) : (x, z) \in \Omega_y\}, \quad \text{where } \Omega_y = \{(x, z) : (x, y, z) \in \Sigma\}.$$

Moreover, the function  $h$  satisfies the following equation

$$(4.1) \quad \frac{\det D^2 h}{(1 + |Dh|^2)^{\frac{3}{2}}} = K (1 + |Dh|^2)^{\frac{1}{2}} = \frac{2\pi}{\mathcal{A}(\Omega)} \langle \vec{e}_3, \vec{n} \rangle (1 + |Dh|^2)^{\frac{1}{2}} = -\frac{2\pi}{\mathcal{A}(\Omega)} h_z.$$

The right hand side of the equation above can be written as  $(2\pi/\mathcal{A})h_v \langle -e_2, v \rangle$ , where  $v$  is the outward normal direction of the level set of  $h$ . Thus, the degenerate Monge-Ampere equation (4.1) has two degenerating factors  $h_v = |Dh|$  and  $\langle -e_2, v \rangle$ . In this section, we study the lower bound for  $\langle -e_2, v \rangle$  which corresponds to the upper bound of  $|\partial_x u|$ , the *partial derivative bound*. Notice that  $|\partial_x u|$  is bounded even on the flat sides.

To obtain the lower bound for  $\langle -e_2, v \rangle$ , we will construct an one parameter family of very wide but short supersolutions  $\varphi_\alpha$  of equation (4.1) with  $\mathcal{A}(\Omega) < 8$ . We will then cut the graph of  $\varphi_\alpha$  so that each of them is contained in a narrow cylinder. By sliding  $\varphi_\alpha$  along the  $z$ -axis we will estimate the partial derivative  $\partial_x u$  at a touching point which will lead to the desired upper bound for  $\partial_x u$  as stated in Theorem 4.4.

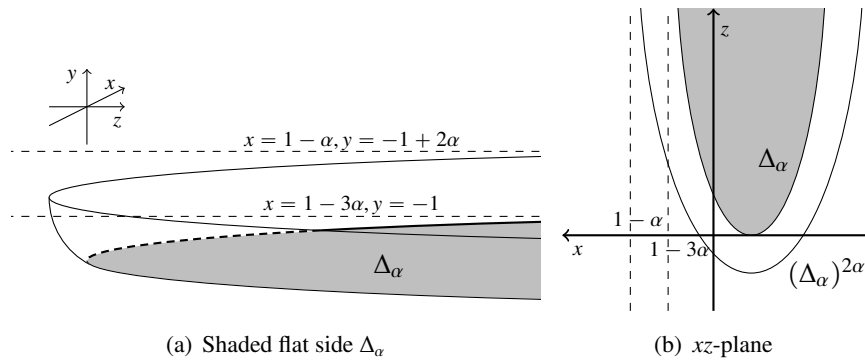


FIGURE 3. Supersolution  $\varphi_\alpha$

**Definition 4.2** (Barrier construction). Given a constant  $\alpha \in (0, 1/6)$ , denote by  $\Delta_\alpha$  the convex set

$$\Delta_\alpha = \left\{ (x, z) \in \mathbb{R}^2 : x \in \left( -\frac{2}{\alpha} + 1 - 3\alpha, 1 - 3\alpha \right), z \geq -\frac{2}{\alpha\pi} \log \cos \left( \frac{\alpha\pi}{2} \left( x - 1 + 3\alpha + \frac{1}{\alpha} \right) \right) \right\}.$$

We denote by  $d_{\Delta_\alpha}(x, z)$  the distance function  $d((x, z), \Delta_\alpha)$ . In particular, if  $(x, z) \in \Delta_\alpha$ , then  $d_{\Delta_\alpha}(x, z) = 0$ . By using  $d_{\Delta_\alpha}(x, z)$ , we define the  $2\alpha$ -extension  $(\Delta_\alpha)^{2\alpha}$  of  $\Delta_\alpha$  by

$$(\Delta_\alpha)^{2\alpha} = \{(x, z) \in \mathbb{R}^2 : d_{\Delta_\alpha}(x, z) \leq 2\alpha\}.$$

Finally, we define the function  $\varphi_\alpha : \text{cl}((\Delta_\alpha)^{2\alpha} \setminus \Delta_\alpha) \rightarrow \mathbb{R}$  by

$$\varphi_\alpha(x, z) = -1 + 2\alpha - \sqrt{4\alpha^2 - d^2(\Delta_\alpha)(x, z)}.$$

This is all shown in Figure 3.

**Lemma 4.3** (Supersolution). *Given a constant  $\alpha \in (0, 1/6)$ , the function  $\varphi_\alpha$  in Definition 4.2 is a convex function satisfying*

$$\frac{\det D^2\varphi}{(1 + |D\varphi|^2)^{\frac{3}{2}}} \leq -\frac{\pi}{4}\varphi_z.$$

*Proof.* For convenience, we let  $\varphi$  and  $d$  denote  $\varphi_\alpha$  and  $d_{\Delta_\alpha}$  respectively. For each point  $p \in \mathbb{R}^2$  with  $d(p) > 0$ , we denote by  $\tau(p)$  and  $\nu(p)$  the tangential and the normal direction of a level set  $L_{d(p)}(d)$  of the distance function  $d$  satisfying  $\langle \tau, e_1 \rangle \geq 0$  and  $\langle \nu, e_2 \rangle \leq 0$ , respectively. Then, we have

$$(4.2) \quad Dd = \nu, \quad d_\nu = |Dd| = 1, \quad d_\tau = 0.$$

We observe  $\nu(p) = \nu(p + \epsilon\nu(p))$  for all  $\epsilon \in \mathbb{R}$  with  $d(p + \epsilon\nu(p)) > 0$ , which implies

$$(4.3) \quad d_{\nu\nu} = d_{\nu\tau} = 0.$$

To derive  $d_{\tau\tau}(p)$ , given a point  $p_0$ , we consider the immersion  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$  satisfying  $d(\gamma(s)) = d(p_0)$  with  $\gamma(0) = p_0$ , where  $s$  is the arc length parameter of the level set,  $\gamma(\mathbb{R}) = L_{d(p_0)}(d)$ . By differentiating  $d(\gamma(s)) = d(p_0)$  twice with respect to  $s$ , we obtain

$$\langle \gamma_s, (D^2d)\gamma_s \rangle + \langle Dd, \gamma_{ss} \rangle = 0$$

We can observe that  $\gamma_s(0) = \tau(p_0)$  and  $\gamma_{ss}(0) = -\kappa(p_0)\nu(p_0)$ , where  $\kappa(p) > 0$  is the curvature of  $L_{d(p)}(d)$  at  $p$ . Hence,  $Dd(\gamma(0)) = \nu(p_0)$  implies  $\langle \tau, (D^2d)\tau \rangle + \langle -\kappa\nu, \nu \rangle = 0$  at  $p_0$ . Thus,

$$(4.4) \quad d_{\tau\tau}(p) = \kappa(p).$$

Hence we can directly derive from (4.2), (4.3) and (4.4) the following holding at each  $p \in \text{cl}((\Delta_\alpha)^{2\alpha} \setminus \Delta_\alpha)$

$$\varphi_\nu = d(4\alpha^2 - d^2)^{-\frac{1}{2}}, \quad \varphi_\tau = 0, \quad \varphi_{\nu\nu} = 4\alpha^2(4\alpha^2 - d^2)^{-\frac{3}{2}}, \quad \varphi_{\nu\tau} = 0, \quad \varphi_{\tau\tau} = \kappa\varphi_\nu.$$

Therefore,  $\varphi$  is a convex function.

Next, combining the equalities above yields

$$(4.5) \quad \frac{\det D^2\varphi}{(1 + |D\varphi|^2)^{\frac{3}{2}}} = \frac{\varphi_{\nu\nu}\varphi_{\tau\tau}}{(1 + \varphi_\nu^2)^{\frac{3}{2}}} = \frac{1}{2\alpha}\varphi_{\tau\tau} = \frac{1}{2\alpha}\kappa\varphi_\nu.$$

Now, we consider the point  $p_0 = p - d(p)v(p) \in \partial\Delta_\alpha$ . Then, the convexity of  $\partial\Delta_\alpha$  leads to

$$(4.6) \quad \kappa(p) \leq \kappa(p_0).$$

We recall that the Grim Reaper curve  $\partial\Delta_\alpha$  is the graph of the convex function  $f_\alpha(x)$  defined by

$$(4.7) \quad f_\alpha(x) = -\frac{2}{\pi\alpha} \log \cos \left( \frac{\alpha\pi}{2} \left( x - 1 + 3\alpha + \frac{1}{\alpha} \right) \right).$$

Hence, at  $x_0$  with  $p_0 = (x_0, f_\alpha(x_0))$ , the following holds

$$\kappa(p_0) = \frac{f_\alpha''(x_0)}{(1 + |f_\alpha'(x_0)|^2)^{\frac{3}{2}}} = \frac{\pi\alpha/2}{(1 + |f_\alpha'(x_0)|^2)^{\frac{1}{2}}} = -\frac{\pi\alpha}{2} \langle v(p_0), e_2 \rangle.$$

Thus,  $v(p_0) = v(p - d(p)v(p)) = v(p)$  implies

$$(4.8) \quad \kappa(p) \leq \kappa(p_0) = -\frac{\pi\alpha}{2} \langle v(p_0), e_2 \rangle = -\frac{\pi\alpha}{2} \langle v(p), e_2 \rangle.$$

Therefore, given a point  $p \in \text{cl}((\Delta_\alpha)^{2\alpha} \setminus \Delta_\alpha)$ , (4.5), (4.6), and (4.8) give the desired result

$$\frac{\det D^2\varphi}{(1 + |D\varphi|^2)^{\frac{3}{2}}} = \frac{1}{2\alpha} \varphi_v \kappa \leq -\frac{\pi}{4} \varphi_v \langle v, e_2 \rangle = -\frac{\pi}{4} (\varphi_v \langle v, e_2 \rangle + \varphi_\tau \langle \tau, e_2 \rangle) = -\frac{\pi}{4} \varphi_z.$$

□

**Theorem 4.4** (Partial derivative bound). *Let  $\Omega$  be an open strictly convex smooth subset of  $\mathbb{R}^2$  which is in additionally axially symmetric and satisfies*

$$[-1, 1] \times [-1, 1] \subset \Omega \subset \left[ -\frac{4}{3}, \frac{4}{3} \right] \times \left[ -\frac{4}{3}, \frac{4}{3} \right].$$

*Then, a solution  $u : \Omega \rightarrow \mathbb{R}$  of (1.2) satisfies*

$$\partial_x u(1 - 5\alpha, -1 + \epsilon) \leq 2 + 3 \cot \frac{\pi\alpha^2}{2},$$

*for all  $\epsilon \in (0, 1/2)$  and  $\alpha \in (0, 1/6)$ .*

*Proof.* Since the domain  $\Omega$  has strictly convex and smooth boundary, Theorem 1.1 implies that the graph  $\Sigma = \{(x, y, u(x, y)) : (x, y) \in \Omega\}$  of the function  $u$  is a strictly convex complete smooth solution of (\*). In addition by Proposition 4.1 the function  $u$  is axially symmetric and we can define a convex set  $\Omega_y$  and a convex function  $h : \Omega_y \rightarrow \mathbb{R}$  by

$$\Omega_y = \{(x, z) : (x, y, z) \in \Sigma\}, \quad \{(x, y, z) \in \Sigma : y \leq 0\} = \{(x, h(x, z), z) : (x, z) \in \Omega_y\}.$$

Then,  $(x, y, u(x, y)) = (x, h(x, z), z)$  implies that the function  $h$  satisfies (4.1). Thus, by the given condition  $\mathcal{A}(\Omega) \leq (8/3)^2 < 8$  and Lemma 4.3, the function  $\varphi_\alpha$  in Definition 4.2 is a supersolution for (4.1).

To construct a barrier  $\Phi_{\epsilon, \alpha}^{t_{\epsilon, \alpha}}$ , we will cut the graph of  $\epsilon + \varphi_\alpha$  by  $\{1 - 4\alpha\} \times \mathbb{R}^2$  (the blue section in Figure 4(a)) and slide it along  $z$ -direction until it touches  $\Sigma$  at a point  $P_0$ . We will show that the contact point  $P_0$  is contained in  $\{1 - 4\alpha\} \times \mathbb{R}^2$ , namely  $P_0$  is a point on the front part of the boundary  $\partial\Phi_{\epsilon, \alpha}^{t_{\epsilon, \alpha}}$  of the barrier. (See the blue curve  $\Gamma_F$  in Figure 4(a)). Then, we will estimate the partial derivative  $\partial_x u$  at  $P_0$  by comparing with the barrier  $\Phi_{\epsilon, \alpha}^{t_{\epsilon, \alpha}}$  at  $P_0$ . After obtaining the bound on  $\partial_x u$  at  $P_0$ , we will use the convexity of the solution  $\Sigma$  the barrier  $\Phi_{\epsilon, \alpha}^{t_{\epsilon, \alpha}}$  to show the desired bound of  $\partial_x u$  at  $(1 - 5\alpha, -1 + \epsilon)$ .

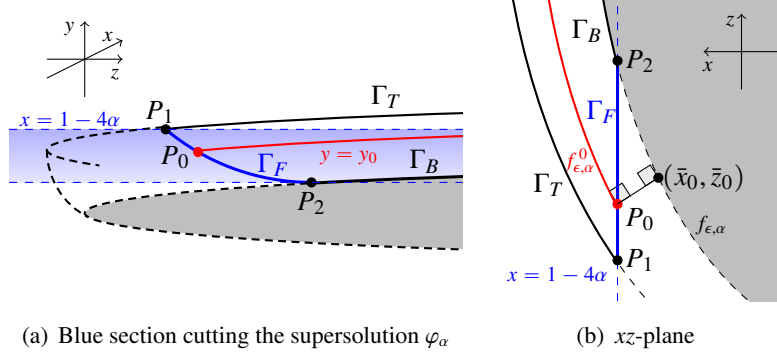


FIGURE 4. Sliding barrier

*Step 1 : Sliding barrier construction.* We denote by  $\Phi_\alpha$  the graph of  $\varphi_\alpha$  in  $[1 - 4\alpha, +\infty) \times \mathbb{R}^2$ ,

$$\Phi_\alpha = \{(x, \varphi_\alpha(x, z), z) : (x, z) \in \text{cl}((\Delta_\alpha)^{2\alpha} \setminus \Delta_\alpha), x \geq 1 - 4\alpha\}.$$

Then, given constants  $\epsilon \in (0, 1/2)$  and  $t \in \mathbb{R}$ , we translate  $\Phi_\alpha$  by  $\epsilon \vec{e}_2 + t \vec{e}_3$ ,

$$\Phi_{\epsilon, \alpha}^t = \{(x, y + \epsilon, z + t) : (x, y, z) \in \Phi_\alpha\}.$$

Notice that definition of  $\varphi_\alpha$  guarantees

$$\Phi_{\epsilon, \alpha}^t \subset [1 - 4\alpha, 1 - \alpha] \times [-1 + \epsilon, -1 + \epsilon + 2\alpha] \times [-t - 2\alpha, +\infty).$$

Hence, there exists a constant  $t_{\epsilon, \alpha} \in \mathbb{R}$  and a point  $P_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$  satisfying

$$(4.9) \quad \Sigma \cap \Phi_{\epsilon, \alpha}^t = \emptyset \quad \text{for } t > t_{\epsilon, \alpha}, \quad P_0 \in \Sigma \cap \Phi_{\epsilon, \alpha}^{t_{\epsilon, \alpha}}.$$

We denote by  $\Delta_{\epsilon, \alpha}$  the projection of  $\Phi_{\epsilon, \alpha}^{t_{\epsilon, \alpha}}$  into the  $xz$ -plane, and consider  $\Phi_{\epsilon, \alpha}^{t_{\epsilon, \alpha}}$  as the graph of a function  $\varphi_{\epsilon, \alpha} : \Delta_{\epsilon, \alpha} \rightarrow \mathbb{R}$

$$\Delta_{\epsilon, \alpha} = \{(x, z) : (x, y, z) \in \Phi_{\epsilon, \alpha}^{t_{\epsilon, \alpha}}\}, \quad \Phi_{\epsilon, \alpha}^{t_{\epsilon, \alpha}} = \{(x, \varphi_{\epsilon, \alpha}(x, z), z) : (x, z) \in \Delta_{\epsilon, \alpha}\}.$$

*Step 2 : Position of the contact point  $P_0$ .* In this step, we will show that the contact point  $P_0$  is contained in the front part  $L_{1-4\alpha}^x(\Phi_{\epsilon, \alpha}^{t_{\epsilon, \alpha}})$  of the boundary  $\partial\Phi_{\epsilon, \alpha}^{t_{\epsilon, \alpha}}$ .

First of all, the contact point  $P_0$  can not be an interior point of  $\Phi_{\epsilon, \alpha}^{t_{\epsilon, \alpha}}$ , because  $\varphi_{\epsilon, \alpha}$  is a supersolution. Thus,  $P_0$  is a point on the boundary  $\partial\Phi_{\epsilon, \alpha}^{t_{\epsilon, \alpha}}$  of  $\Phi_{\epsilon, \alpha}^{t_{\epsilon, \alpha}}$ . We observe that the boundary  $\partial\Phi_{\epsilon, \alpha}^{t_{\epsilon, \alpha}}$  can be decomposed into the top  $\Gamma_T$ , bottom  $\Gamma_B$ , and front  $\Gamma_F$  boundary as following

$$(4.10) \quad \partial\Phi_{\epsilon, \alpha}^{t_{\epsilon, \alpha}} = \Gamma_T \cup \Gamma_B \cup \Gamma_F, \quad \Gamma_T = L_{-1+\epsilon+2\alpha}^y(\Phi_{\epsilon, \alpha}^{t_{\epsilon, \alpha}}), \quad \Gamma_B = L_{-1+\epsilon}^y(\Phi_{\epsilon, \alpha}^{t_{\epsilon, \alpha}}), \quad \Gamma_F = L_{1-4\alpha}^x(\Phi_{\epsilon, \alpha}^{t_{\epsilon, \alpha}}).$$

We denote by  $P_1$  and  $P_2$  the end point of the top  $\Gamma_T$  and the bottom  $\Gamma_B$  boundary, respectively

$$(4.11) \quad \begin{aligned} P_1 &= (x_1, y_1, z_1) = \Gamma_T \cap \Gamma_F = (1 - 4\alpha, -1 + \epsilon + 2\alpha, z_1), \\ P_2 &= (x_2, y_2, z_2) = \Gamma_B \cap \Gamma_F = (1 - 4\alpha, -1 + \epsilon, z_2). \end{aligned}$$

On the other hand (4.9) gives  $\text{cl}(\Delta_{\epsilon,\alpha}) =: \Delta_{\epsilon,\alpha} \subset \text{Int}(\Omega_y)$  and  $h(x, z) \leq \varphi_{\epsilon,\alpha}(x, z)$  on  $\Delta_{\epsilon,\alpha}$ . Also, we have  $|D\varphi_{\epsilon,\alpha}| = +\infty$  on  $\Gamma_T$ . Hence, if  $P_0 \in (\Gamma_T \setminus \{P_1\})$ , then  $|Dh| = +\infty$  holds at  $P_0$  by  $h \leq \varphi_{\epsilon,\alpha}$ , which contradicts to  $\text{cl}(\Delta_{\epsilon,\alpha}) \subset \text{Int}(\Omega_y)$ . Thus,

$$P_0 \notin (\Gamma_T \setminus \{P_1\})$$

Moreover, we have  $|D\varphi_{\epsilon,\alpha}| = 0$  on  $\Gamma_B$ . Thus, if  $P_0 \in (\Gamma_B \setminus \{P_2\})$ , then  $|Dh| = 0$  holds at  $P_0$ . However,  $\Sigma$  is a strictly convex complete surface, which means  $|Dh| \neq 0$ . Therefore,

$$P_0 \notin (\Gamma_B \setminus \{P_2\})$$

Hence, by (4.10) and (4.11),  $P_0$  is a point on the front boundary  $\Gamma_F$

$$(4.12) \quad P_0 = (x_0, y_0, z_0) = (1 - 4\alpha, y_0, z_0) \in \Gamma_F.$$

*Step 3 : Distance between  $P_0$  and  $P_2$ .* In this step, we will estimate  $z_2 - z_0$  in terms of  $\alpha$ .

We recall that the Grim reaper curve  $\partial\Delta_\alpha$  is the graph of the function  $f_\alpha(x)$  defined by (4.7) and  $\Delta_{\epsilon,\alpha}$  is a subset of  $\bar{\Delta}_{\epsilon,\alpha} =: \Delta_\alpha + t_{\epsilon,\alpha}e_z$ . Hence,  $\partial\bar{\Delta}_{\epsilon,\alpha}$  is the graph of the function  $f_{\epsilon,\alpha}$  defined by

$$(4.13) \quad f_{\epsilon,\alpha}(x) = t_{\epsilon,\alpha} + f_\alpha(x) = t_{\epsilon,\alpha} - \frac{2}{\pi\alpha} \log \cos \left( \frac{\alpha\pi}{2} \left( x - 1 + 3\alpha + \frac{1}{\alpha} \right) \right).$$

By definition of  $\varphi_\alpha$ , there exists a unique point  $(\bar{x}_0, \bar{z}_0) \in \partial\Delta_{\epsilon,\alpha}$  such that

$$(4.14) \quad d((x_0, z_0), \bar{\Delta}_{\epsilon,\alpha}) = d((x_0, z_0), (\bar{x}_0, \bar{z}_0)) \leq 2\alpha.$$

We know  $x_0 = x_2 = 1 - 4\alpha$  by (4.11) and (4.12). Hence, for all  $x \in [\bar{x}_0, x_2]$ , we can derive from (4.13) the following inequality

$$(4.15) \quad f'_{\epsilon,\alpha}(\bar{x}_0) \leq f'_{\epsilon,\alpha}(x) \leq f'_{\epsilon,\alpha}(x_2) = \tan \left( \frac{\pi\alpha}{2} \left( x_0 - 1 + 3\alpha + \frac{1}{\alpha} \right) \right) = \cot \frac{\pi\alpha^2}{2}.$$

Therefore, combining (4.14) and (4.15) yields

$$(4.16) \quad z_2 - z_0 \leq (\bar{z}_0 - z_0) + (z_2 - \bar{z}_0) \leq 2\alpha + (z_6 - \bar{z}_0) \leq 2\alpha + \int_{\bar{x}_0}^{x_2} f'_{\epsilon,\alpha}(x) dx \leq 2\alpha + 2\alpha \cot \frac{\pi\alpha^2}{2}.$$

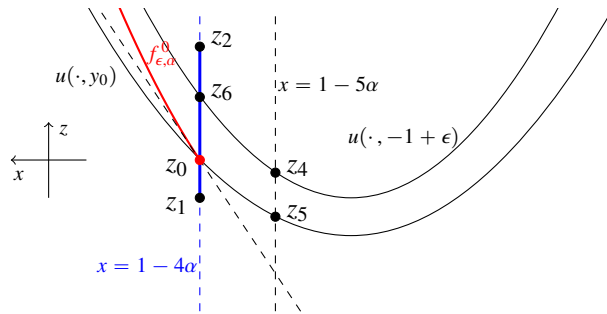


FIGURE 5. Level sets of the solution  $h$

*Step 4 : Partial derivative  $\partial_x u$  bound at the contact point  $P_0$ .* We can consider the level set  $L_{y_0}(\varphi_{\epsilon,\alpha})$  as the graph of a convex function  $f_{\epsilon,\alpha}^0 : [1 - 4\alpha, 1 - \alpha] \rightarrow \mathbb{R}$ , namely  $L_{y_0}(\varphi_{\epsilon,\alpha}) = \{(x, f_{\epsilon,\alpha}^0(x)) : x \in$

$[1 - 4\alpha, 1 - \alpha]\}$ . Then, by the definitions of  $\varphi_\alpha$  and  $(\bar{x}_0, \bar{z}_0)$ , we have  $(f_{\epsilon,\alpha}^0)'(x_0) = f'_{\epsilon,\alpha}(\bar{x}_0)$ . Thus, (4.15) yields the bound

$$(f_{\epsilon,\alpha}^0)'(x_0) \leq \cot(\pi\alpha^2/2).$$

On the other hand, (4.9) implies  $L_{y_0}(\varphi_{\epsilon,\alpha}) < L_{y_0}(h)$ , namely  $u(x, y_0) \leq f_{\epsilon,\alpha}^0(x)$  holds for all  $x \in [1 - 4\alpha, 1 - \alpha]$ . Therefore,

$$(4.17) \quad \partial_x u(x_0, y_0) \leq (f_{\epsilon,\alpha}^0)'(x_0) \leq \cot(\pi\alpha^2/2).$$

*Step 5 : Partial derivative  $\partial_x u$  bound at the given point.* We define the points  $P_3, P_4, P_5$  on  $\Sigma$  by

$$\begin{aligned} P_3 &= (x_3, y_3, z_3) =: (1 - 5\alpha, y_0, u(1 - 5\alpha, y_0)), \\ P_4 &= (x_4, y_4, z_4) =: (1 - 5\alpha, -1 + \epsilon, u(1 - 5\alpha, -1 + \epsilon)), \\ P_5 &= (x_5, y_5, z_5) =: (1 - 4\alpha, -1 + \epsilon, u(1 - 4\alpha, -1 + \epsilon)). \end{aligned}$$

Since we know  $P_0, P_3 \in L_{y_0}^y(\Sigma)$ , the inequality (4.17) and the convexity of  $u$  give

$$z_0 - z_3 = \int_{x_3}^{x_0} \partial_x u(x, y_0) dx \leq \int_{x_3}^{x_0} \partial_x u(x_0, y_0) dx = \alpha (\partial_x u(x_0, y_0)) \leq \alpha \cot \frac{\pi\alpha^2}{2}.$$

By adding (4.16) and the inequality above, we obtain

$$(4.18) \quad z_2 - z_3 \leq 2\alpha + 3\alpha \cot(\pi\alpha^2/2).$$

On the other hand, (4.9) implies that  $L_{-1+\epsilon}(\varphi_{\epsilon,\alpha}) < L_{-1+\epsilon}(h)$ . Therefore,  $x_2 = x_5 = 1 - 4\alpha$ ,  $(x_2, z_2) \in L_{-1+\epsilon}(\varphi_{\epsilon,\alpha})$ , and  $(x_5, z_5) \in L_{-1+\epsilon}(h)$  guarantee

$$z_2 = \varphi_{\epsilon,\alpha}(x_2) \geq u(x_2, -1 + \epsilon) = u(x_5, -1 + \epsilon) = z_5.$$

Also, the convexity and symmetry of  $\Sigma$  give that

$$z_3 \leq z_4$$

Thus, subtracting the inequalities above yields  $z_5 - z_4 \leq z_2 - z_3$ . Applying (4.18), we have

$$z_5 - z_4 \leq 2\alpha + 3\alpha \cot(\pi\alpha^2/2).$$

Hence, the desired result follows by the following computation

$$z_5 - z_4 = \int_{x_4}^{x_5} \partial_x u(x, -1 + \epsilon) dx \geq \int_{x_4}^{x_5} \partial_x u(x_4, -1 + \epsilon) dx = \alpha (\partial_x u(1 - 5\alpha, -1 + \epsilon)).$$

□

## 5. DISTANCE FROM THE TIP TO FLAT SIDES

Let  $\Sigma$  be the translating solution to the Gauss curvature flow over the square  $\Omega$  as in Theorem 1.2. In this final section we will show that this solution has flat sides, as stated in Theorem 5.2 below. To this end, we will study the distance from the tip of a solution  $\Sigma$  to each point on the *free boundary*, that is the boundary of the flat sides. To estimate this distance one needs to establish a *gradient bound* for solutions to the equation (1.2) at a *certain point* near the flat sides. Since the gradient bound depends on the global structure of  $\Omega$ , we will establish an integral estimate by deriving a separation of variables structure from (1.2) as in the proof of the following Lemma.

**Lemma 5.1** (Gradient bound). *Let  $\Omega$  satisfy the conditions in Theorem 4.4 and let  $u$  be a solution of (1.2) on  $\Omega$ . Assume that  $[a, b] \times [-1, -1 + \sigma] \subset \Omega$ , for some constants  $a, b, \sigma \in (0, 1)$ . Then, there exists a point  $x_0 \in [a, b]$  satisfying*

$$-\partial_y u(x_0, -1 + \sigma) \leq \sqrt{\frac{2M}{\pi\sigma(b-a)}},$$

where  $M = \sup_{y \in (0, \sigma)} \partial_x u(b, -1 + y)$ .

*Proof.* Since the domain  $\omega$  satisfies the assumptions of Theorem 4.4, we have  $\mathcal{A} \leq 64/9$ . Hence, the following inequality holds

$$\frac{u_{yy}u_{xx}}{(1+u_y^2)^{\frac{3}{2}}} \geq \frac{u_{yy}u_{xx} - u_{xy}^2}{(1+u_x^2 + u_y^2)^{\frac{3}{2}}} = \frac{\det D^2u}{(1+|Du|^2)^{\frac{3}{2}}} = \frac{2\pi}{\mathcal{A}} \geq \frac{\pi}{4}.$$

which combined with Holder inequality yields

$$\left( \int_a^b \frac{u_{yy}}{(1+u_y^2)^{\frac{3}{2}}} dx \right) \left( \int_a^b u_{xx} dx \right) \geq \left( \int_a^b (\pi/4)^{\frac{1}{2}} dx \right)^2 = \frac{\pi}{4}(a-b)^2.$$

Since  $a \geq 0$  and Proposition 4.1 guarantee that  $u_x(\cdot, a) \geq 0$ , we have

$$\int_a^b u_{xx} dx = u_x(y, b) - u_x(y, a) \leq u_x(y, b) \leq M.$$

Hence,

$$\frac{1}{b-a} \int_a^b \int_{-1}^{-1+\sigma} \frac{u_{yy}}{(1+u_y^2)^{\frac{3}{2}}} dy dx \geq \frac{\pi\sigma}{4M}(b-a).$$

Therefore, there exists a constant  $x_0 \in [a, b]$  satisfying

$$\int_{-1}^{-1+\sigma} \frac{u_{yy}}{(1+u_y^2)^{\frac{3}{2}}}(x_0, \cdot) dy \geq \frac{\pi\sigma}{4M}(b-a).$$

Moreover, by  $|u_y| \leq (1+u_y^2)^{\frac{1}{2}}$ , we have

$$\int_{-1}^{-1+\sigma} \frac{u_{yy}}{(1+u_y^2)^{\frac{3}{2}}}(x_0, \cdot) dy = \frac{u_y}{(1+u_y^2)^{\frac{1}{2}}}(x_0, \cdot) \Big|_{-1}^{-1+\sigma} \leq \frac{u_y}{(1+u_y^2)^{\frac{1}{2}}}(x_0, -1 + \sigma) + 1.$$

Hence, at  $(x_0, -1 + \sigma)$ , the following holds

$$\frac{|u_y|}{(1 + u_y^2)^{\frac{1}{2}}}(x_0, -1 + \sigma) = \frac{-u_y}{(1 + u_y^2)^{\frac{1}{2}}}(x_0, -1 + \sigma) \leq 1 - \frac{\pi\sigma}{4M}(b - a)$$

implying the bound

$$1 + u_y^{-2}(x_0, -1 + \sigma) \geq \left(1 - \frac{\pi\sigma}{4M}(b - a)\right)^{-2} \geq \left(1 + \frac{\pi\sigma}{4M}(b - a)\right)^2 \geq 1 + \frac{\pi\sigma}{2M}(b - a).$$

Solving this last inequality for  $-u_y(x_0, -1 + \sigma) \geq 0$  leads to the desired result.  $\square$

The distance between the tip of the translating solution  $\Sigma$  over the square and its flat sides is estimated in the following result.

**Theorem 5.2** (Distance between the tip and flat sides). *Let  $\Omega$  satisfy the conditions in Theorem 4.4 and let  $u$  be a solution of (1.2). Given  $\alpha \in (0, 1/6)$ , there exists a constant  $C > 0$  satisfying*

$$u(1 - 6\alpha, -1) - u(0, 0) \leq 6\left(1 + \frac{1}{\alpha^2}\right).$$

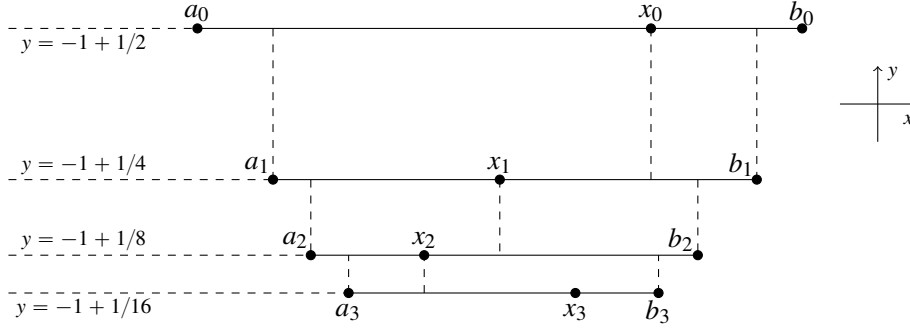


FIGURE 6. Converging points on domain  $\Omega$

*Proof.* We begin by setting  $a_0 = 1 - 6\alpha$ ,  $b_0 = 1 - 5\alpha$ ,  $\sigma_0 = \frac{1}{2}$ , and  $M = \sup_{y \in (0, 1/2)} \partial_x u(b_0, -1 + y)$ . Then, by Lemma 5.1, there exists a point  $x_0 \in [a_0, b_0]$  satisfying

$$(5.1) \quad -\partial_y u(x_0, -1 + 1/2) \leq 2(M/\pi\alpha)^{\frac{1}{2}}.$$

We choose an interval  $[a_1, b_1]$  satisfying  $x_0 \in [a_1, b_1] \subset [a_0, b_0]$  and  $b_1 - a_1 = 2^{-1/3}\alpha$ . Then, for  $\sigma_1 = 2^{-2}$ , we have  $\sup_{y \in (0, \sigma_1)} \partial_x u(b_1, -1 + y) \leq M$ . Hence, Lemma 5.1 gives a point  $x_1 \in [a_1, b_1]$  satisfying

$$-\partial_y u(x_1, -1 + 1/2^2) \leq 2^{1+\frac{2}{3}}(M/\pi\alpha)^{\frac{1}{2}}.$$

By setting  $\sigma_n = 2^{-1-n}$ , we can inductively choose intervals  $[a_n, b_n]$  satisfying  $x_{n-1} \in [a_n, b_n] \subset [a_{n-1}, b_{n-1}]$  and  $b_n - a_n = 2^{-n/3}\alpha$  so that we obtain a point  $x_n \in [a_n, b_n]$  satisfying

$$-\partial_y u(x_n, -1 + 1/2^{n+1}) \leq 2^{1+\frac{2n}{3}}(M/\pi\alpha)^{\frac{1}{2}}.$$



Then, integrating along  $y$  yields

$$\left| u(x_n, -1 + \frac{1}{2^{n+1}}) - u(x_n, -1 + \frac{1}{2^n}) \right| \leq \int_{-1+1/2^{n+1}}^{-1+1/2^n} -\partial_y u(x_n, y) dy \leq 2^{-n/3} (M/\pi\alpha)^{\frac{1}{2}}.$$

On the other hand,  $x_{n-1}, x_n \in [a_n, b_n]$  and  $a_n - b_n = 2^{-n/3}\alpha$  imply

$$\left| u(x_n, -1 + \frac{1}{2^n}) - u(x_{n-1}, -1 + \frac{1}{2^n}) \right| = \left| \int_{x_{n-1}}^{x_n} \partial_x u(x, -1 + \frac{1}{2^n}) dx \right| \leq 2^{-\frac{n}{3}} M\alpha.$$

Therefore,

$$\left| u(x_n, -1 + \frac{1}{2^{n+1}}) - u(x_{n-1}, -1 + \frac{1}{2^n}) \right| \leq 2^{-\frac{n}{3}} ((M/\pi\alpha)^{\frac{1}{2}} + M\alpha).$$

By definition of  $x_n$ , the sequence  $\{x_n\}_{n \in \mathbb{N}}$  converges to a point  $\bar{x} \in [a_0, b_0]$ . Hence, we can sum up the inequality above for all  $n \in \mathbb{N}$  so that we have

$$(5.2) \quad u(\bar{x}, -1) - u(x_0, -\frac{1}{2}) \leq \sum_{n=1}^{\infty} 2^{-\frac{n}{3}} ((M/\pi\alpha)^{\frac{1}{2}} + M\alpha) \leq 4((M/\pi\alpha)^{\frac{1}{2}} + M\alpha).$$

Next, we consider the linear function

$$f(x, y) = (\partial_x u(x_0, -1/2)) (x - x_0) + (\partial_y u(x_0, -1/2)) (y + 1/2) + u(x_0, -1/2)$$

whose graph is the tangent hyperplane of  $\Sigma$  at  $(x_0, -1/2, u(x_0, -1/2))$ . Then, the convexity of  $\Sigma$  gives  $f(0, 0) \leq u(0, 0)$ . Hence, (5.1) and definition of  $M$  show

$$u(x_0, -\frac{1}{2}) - u(0, 0) \leq f(x_0, -\frac{1}{2}) - f(0, 0) \leq (M/\pi\alpha)^{\frac{1}{2}} + M.$$

Thus, Theorem 4.4, (5.2), and the inequality give

$$u(\bar{x}, -1) - u(0, 0) \leq 5\left(\frac{M}{\pi\alpha}\right)^{\frac{1}{2}} + 2M \leq 5\left(\frac{2}{\pi\alpha} + \frac{3}{\pi\alpha} \cot \frac{\pi\alpha^2}{2}\right)^{\frac{1}{2}} + 2\left(2 + 3 \cot \frac{\pi\alpha^2}{2}\right).$$

Applying  $1/\alpha \geq 6$  and  $\cot(\pi\alpha^2/2) \leq 2/(\pi\alpha^2)$ , we have

$$u(\bar{x}, -1) - u(0, 0) \leq 5\left(\frac{2}{\pi\alpha} + \frac{6}{\pi^2\alpha^3}\right)^{\frac{1}{2}} + \frac{12}{\pi\alpha^2} + 4 \leq 6\left(1 + \frac{1}{\alpha^2}\right).$$

Finally, combining Proposition 4.1, the convexity of  $u$ , and  $\bar{x} \geq b_0 = 1 - 6\alpha \geq 0$  yields  $u(\bar{x}, -1) \geq u(1 - 6\alpha, -1)$ , which leads to the desired result.  $\square$

We will now give the proof of the main Theorem 1.2. This readily follows from the following result.

**Theorem 5.3** (Existence of flat sides). *Let  $\Omega = (-1, 1) \times (-1, 1)$  and  $u$  be a solution of (1.2) on  $\Omega$ . Then, there exists a function  $\bar{u} : (\partial\Omega \setminus V) \rightarrow \mathbb{R}$  satisfying*

$$\bar{u}(x_0) = \lim_{x \rightarrow x_0} u(x).$$

*Also, the corresponding complete solution  $\Sigma$  of (\*) is a convex surface of class  $C_{loc}^{1,1}$ , and  $u$  satisfies*

$$\lim_{x \rightarrow V} u(x) = +\infty.$$

*Proof.* Let  $\{\Omega_n\}_{n \in \mathbb{N}}$  be a sequence of sets satisfying the conditions in Theorem 4.4, and let  $\{u_n\}_{n \in \mathbb{N}}$  and  $\{\Sigma_n\}_{n \in \mathbb{N}}$  be the sequence of corresponding solutions of (1.2) with  $u_n(0, 0) = 0$  and their graphs, respectively. We denote the convex hull of  $\Sigma_n$  by  $E_n = \{tX + (1 - t)Y : X, Y \in \Sigma_n, t \in [0, 1]\}$ , and define a convex body  $E$  by

$$E = \bigcap_{n \in \mathbb{N}} E_n.$$

Observe that  $E$  is not an empty set, because Theorem 5.2 gives that  $(1 - 6\alpha, -1, 6(1 + \alpha^{-2})) \in E_n$  which means that  $(1 - 6\alpha, -1, 6(1 + \alpha^{-2})) \in E$ . Now, we denote by  $\Sigma$  the boundary of  $E$ . Then,  $\Sigma$  is naturally a convex complete and non-compact surface, since  $\{(0, 0, t) : t \geq 0\} \subset E_n$  implies  $\{(0, 0, t) : t \geq 0\} \subset E$ .

We define  $u : \Omega \rightarrow \mathbb{R}$  by  $(x, y, u(x, y)) \in \Sigma$ . Then, the symmetry of  $\Omega_n$ , the convexity of  $E$ , and  $(1 - 6\alpha, -1, 6(1 + \alpha^{-2})) \in E$  guarantees that there exists a function  $\bar{u} : (\partial\Omega \setminus V) \rightarrow \mathbb{R}$  satisfying

$$\bar{u}(x_0) = \lim_{x \rightarrow x_0} u(x).$$

Moreover, Theorem 3.1 shows the local  $C^{1,1}$  regularity of  $\Sigma$ . Now, we assume that there exists a point  $(1, 1, t_0)$  in  $E$ , then the mean curvature of  $\Sigma$  attains  $+\infty$  at  $(1, 1, t)$  for  $t > t_0$ . This contradicts to the local  $C^{1,1}$  regularity of  $\Sigma$ . Hence, we have  $\Sigma \cap (V \times \mathbb{R}) = \emptyset$ , namely  $u$  satisfies

$$\lim_{x \rightarrow V} u(x) = +\infty.$$

□

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